In [1]:

```
import numpy as np
from scipy import optimize
from scipy import interpolate

import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
plt.style.use('seaborn-whitegrid')
colors = [x['color'] for x in plt.style.library['seaborn']['axes.prop_cycle']]
from matplotlib import cm
```

Linear regression

In [2]:

```
def DGP(N):
    """ data generating process

    Args:

        N (int): number of observations

    Returns:

        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    """

    # a. independent variables
    x1 = np.random.normal(0,1,size=N)
    x2 = np.random.normal(0,1,size=N)

    # b. errors
    eps = np.random.normal(0,1,size=N)

    extreme = np.random.uniform(0,1,size=N)
    eps[extreme < 0.05] += np.random.normal(-5,1,size=N)[extreme < 0.05]
    eps[extreme > 0.95] += np.random.normal(5,1,size=N)[extreme > 0.95]

    # d. depenent variable
    y = 0.1 + 0.3*x1 + 0.5*x2 + eps

    return x1, x2, y
```

In [3]:

```
np.random.seed(2020)
x1,x2,y = DGP(10000)
```

Question 1

In [4]:

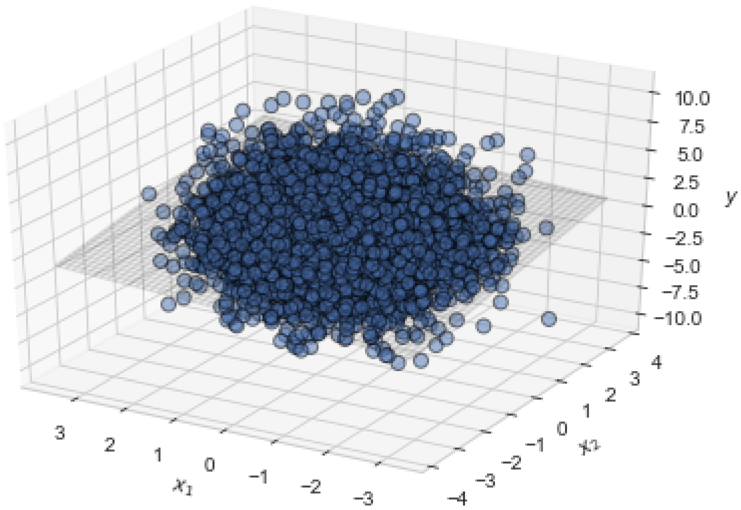
```
def OLS_estimate(x1,x2,y):
    """ compute OLS estimates using matrix algebra

    Args:

        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    Returns:

        betas (ndrarray): estimates


    """

    X = np.vstack((np.ones(x1.size),x1,x2)).T
    betas = (np.linalg.inv(X.T@X)@X.T)@y
    return betas

betas = OLS_estimate(x1,x2,y)
for i,beta in enumerate(betas):
    print(f'beta{i} = {beta:.4f}')
```

```
beta0 = 0.0957
beta1 = 0.2929
beta2 = 0.5033
```

Question 2

In [5]:

```
fig = plt.figure()
ax = fig.add_subplot(1,1,1,projection='3d')

# a. predicted
x1v = np.linspace(x1.min(),x1.max(),100)
x2v = np.linspace(x2.min(),x2.max(),100)
x1g, x2g = np.meshgrid(x1v,x2v,indexing='ij')
yhat = betas[0] + betas[1]*x1g + betas[2]*x2g
ax.plot_wireframe(x1g,x2g,yhat,color='black',alpha=0.1);

# b. scatter
ax.scatter(x1,x2,y,s=50,edgecolor='black',facecolor=colors[0],alpha=0.5);

# c. details
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$y$')
ax.invert_xaxis()
fig.tight_layout()
```



Question 3

In [6]:

```
def OLS_objective(betas,x1,x2,y):
    """ OLS objective

    Args:

        betas (ndarray): current guess on parameters
        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    Returns:

        ressum (ndrarray): sum of squared residuals

    """

    yhat = betas[0] + betas[1]*x1 + betas[2]*x2
    ressum = np.sum((yhat-y)**2)
    return ressum

def OLS_estimate_alt(x1,x2,y):
    """ compute OLS estimates using numerical optimizer

    Args:

        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    Returns:

        betas (ndarray): parameter estimates

    """

    betas0 = np.array([0.1,0.3,0.5])
    sol = optimize.minimize(OLS_objective,betas0,args=(x1,x2,y),method='Nelder-Mead')
    betas = sol.x
    return betas
```

In [7]:

```
betas = OLS_estimate_alt(x1,x2,y)
for i,beta in enumerate(betas):
    print(f'beta{i} = {beta:.4f}')
```

```
beta0 = 0.0957
beta1 = 0.2929
beta2 = 0.5033
```

Question 4

In [8]:

```
def LAD_objective(betas,x1,x2,y):
    """ LAD objective

    Args:

        betas (ndarray): current guess on parameters
        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    Returns:

        ressum (ndrarray): sum of absolute residuals

    """

    yhat = betas[0] + betas[1]*x1 + betas[2]*x2
    ressum = np.sum(np.abs(yhat-y))
    return ressum

def LAD_estimate(x1,x2,y):
    """ compute LAD estimates using numerical optimizer

    Args:

        x1 (ndarray): independent variable x1
        x2 (ndarray): independent variable x2
        y (ndarray): dependent varialbe y

    Returns:

        betas (ndarray): parameter estimates

    """

    betas0 = np.array([0.1,0.3,0.5])
    sol = optimize.minimize(LAD_objective,betas0,args=(x1,x2,y),method='Nelder-Mead')
    betas = sol.x
    return betas
```

In [9]:

```
betas = LAD_estimate(x1,x2,y)
for i,beta in enumerate(betas):
    print(f'beta{i} = {beta:.4f}')
```

```
beta0 = 0.0921
beta1 = 0.3074
beta2 = 0.5116
```

Question 5

In [10]:

```
# a. setup
K = 5000
N = 50

# b. allocate
betas_OLS = np.empty((K,3))
betas_LAD = np.empty((K,3))

# c. estimate
for k in range(K):
    x1,x2,y = DGP(N)
    betas_OLS[k,:] = OLS_estimate(x1,x2,y)
    betas_LAD[k,:] = LAD_estimate(x1,x2,y)
```

Mean and std.:

In [11]:

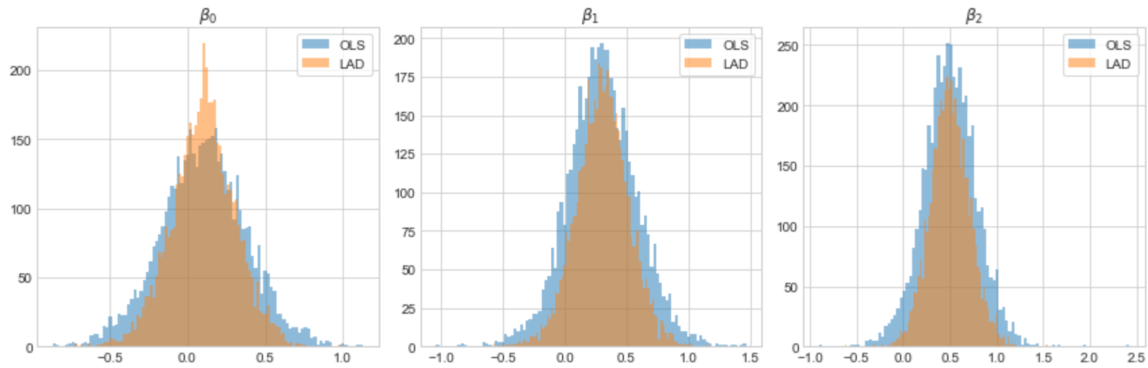
```
for i in range(3):
    print(f'beta{i}')
    print(f' OLS: mean = {np.mean(betas_OLS[:,i]):.4f}, std. = {np.std(betas_OLS[:,i]):
.4f}')
    print(f' LAD: mean = {np.mean(betas_LAD[:,i]):.4f}, std. = {np.std(betas_LAD[:,i]):
.4f}')
```

```
beta0
 OLS: mean = 0.1026, std. = 0.2749
 LAD: mean = 0.1025, std. = 0.2032
beta1
 OLS: mean = 0.3019, std. = 0.2844
 LAD: mean = 0.3027, std. = 0.2134
beta2
 OLS: mean = 0.5003, std. = 0.2860
 LAD: mean = 0.5008, std. = 0.2116
```

Histograms:

In [12]:

```
fig = plt.figure(figsize=(12,4))

for i in range(3):

    ax = fig.add_subplot(1,3,i+1)
    ax.hist(betas_OLS[:,i],bins=100,alpha=0.5,label='OLS');
    ax.hist(betas_LAD[:,i],bins=100,alpha=0.5,label='LAD');
    ax.set_title(f'$\\beta_{{{i}}}$')
    ax.legend(frameon=True)

fig.tight_layout()
```



Durable consumption

In [13]:

```
# a. parameters
rho = 2
alpha = 0.8
chi = 0.9
beta = 0.96
r = 0.04
Delta = 0.25

# b. grids
d_vec = np.linspace(1e-8,5,100)
m1_vec = np.linspace(1e-8,10,100)
m2_vec = np.linspace(0.5,10,100)
```

Question 1

In [14]:

```
def utility(c,d,x,rho,alpha,chi):
    """ utility

    Args:

        c (float): non-durable consumption
        d (float): pre-commited durable consumption
        x (float): extra durable consumption
        rho (float): CRRA parameter
        alpha (float): utility weight on non-durable consumption
        chi (float): scala factor for extra durable consumption

    Returns:

        (float): utility of consumption

    """

    return (c**alpha*(d+chi*x)**(1-alpha))**(1-rho)/(1-rho)

def v2(c,m2,d,rho,alpha,chi):
    """ value of choice in period 2

    Args:

        c (float): non-durable consumption
        m2 (float): cash-on-hand in beginning of period 2
        d (float): pre-commited durable consumption
        x (float): extra durable consumption
        rho (float): CRRA parameter
        alpha (float): utility weight on non-durable consumption
        chi (float): scala factor for extra durable consumption

    Returns:

        (float): value-of-choice

    """

    x = m2-c
    return utility(c,d,x,rho,alpha,chi)
```

In [15]:

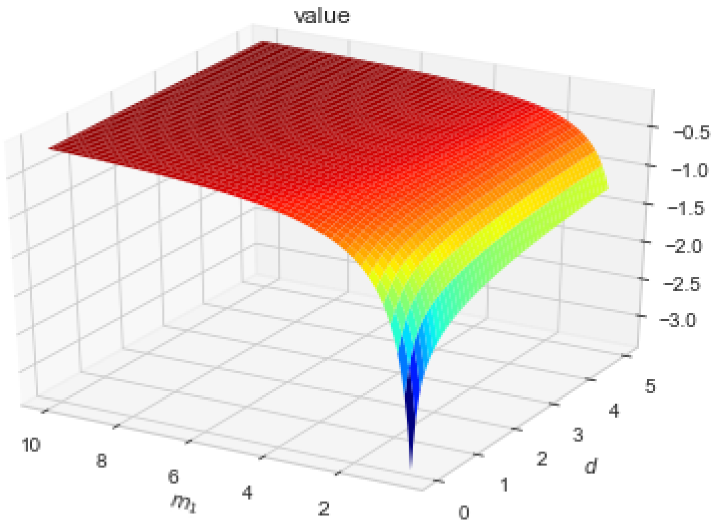
```
def solve_period_2(m2_vec,d_vec,rho,alpha,chi):
    """ solve consumer problem in period 2

    Args:

        m2 (ndarray): vector of cash-on-hand in beginning of period 2
        d (ndarray): vector of pre-commited durable consumption
        d (float): pre-commited durable consumption
        x (float): extra durable consumption
        rho (float): CRRA parameter
        alpha (float): utility weight on non-durable consumption
        chi (float): scala factor for extra durable consumption

    Returns:

        v2_mat (ndarray): value function in period 2
        c_ast_mat (ndarray): consumption function
        x_ast_mat (ndarray): implied extra durable consumption function

    """

    # a. allocate
    v2_mat = np.empty((m2_vec.size,d_vec.size))
    c_ast_mat = np.empty((m2_vec.size,d_vec.size))
    x_ast_mat = np.empty((m2_vec.size,d_vec.size))

    # b. loop over states
    for i,m2 in enumerate(m2_vec):
        for j,d in enumerate(d_vec):

            # i. objective
            obj = lambda c: -v2(c,m2,d,rho,alpha,chi)

            # ii. initial value (consume half)
            x0 = m2/2

            # iii. optimizer
            result = optimize.minimize_scalar(obj,x0,method='bounded',bounds=[1e-8,m2])

            # iv. save
            v2_mat[i,j] = -result.fun
            c_ast_mat[i,j] = result.x
            x_ast_mat[i,j] = m2-c_ast_mat[i,j]

    return v2_mat,c_ast_mat,x_ast_mat
```

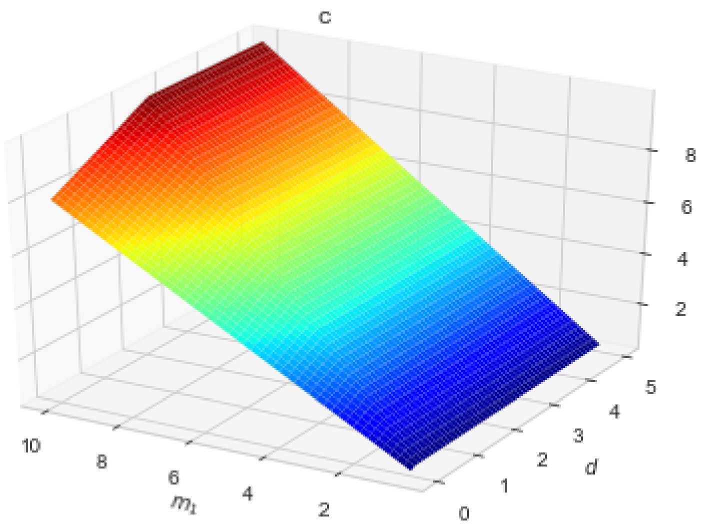
In [16]:

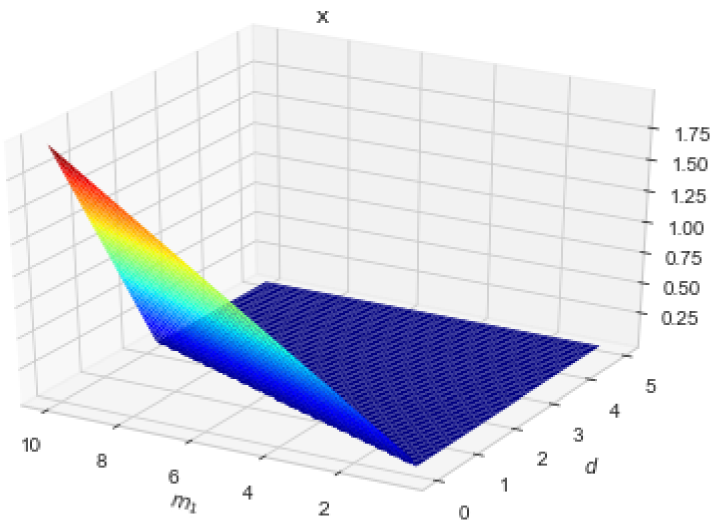
```
v2_mat,c_ast_mat,x_ast_mat = solve_period_2(m2_vec,d_vec,rho,alpha,chi)
```

In [17]:

```
for ystr,y in [('value',v2_mat),('c',c_ast_mat),('x',x_ast_mat)]:

    fig = plt.figure()
    ax = fig.add_subplot(1,1,1,projection='3d')

    # a. value function
    m2g,dg = np.meshgrid(m2_vec,d_vec,indexing='ij')
    ax.plot_surface(m2g,dg,y,cmap=cm.jet)

    # b. details
    ax.set_title(ystr)
    ax.set_xlabel('$m_1$')
    ax.set_ylabel('$d$')
    ax.set_zlabel('')
    ax.invert_xaxis()
    fig.tight_layout()
```


value
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
10
8
6
4
2
m_1
0
1
2
3
4
5
d


c
8
6
4
2
10
8
6
4
2
m_1
0
1
2
3
4
5
d



Questions 2

In [18]:

```
v2_interp = interpolate.RegularGridInterpolator([m2_vec,d_vec], v2_mat,
                                                bounds_error=False,fill_value=None)
```

In [19]:

```
def w(a,d,r,Delta,v2_interp):
    """ post-decision value function in period 1

    Args:

        a (float): end-of-period asset
        d (float): pre-commited durable consumption
        r (float): return on savings
        Delta (float): income risk scale factor
        v2_interp (RegularGridInterpolator): interpolator for value function in period
 2

    Returns:

        (ndarray): discounted post-decision value

    """

    # a. initialize
    w = 0
    y = np.array([1-Delta,1,1+Delta])

    # b. loop over shocks
    for i in range(3):
        m2_now = (1+r)*a + y[i]
        v2_now = v2_interp([m2_now,d])[0]
        w += 1/3*v2_now

    return beta*w
```

In [20]:

```
def v1(d,m1,beta,r,Delta,v2_interp):
    """ post-decision value function in period 1

    Args:

        d (float): pre-commited durable consumption
        m1 (float): cash-on-hand in the beginning of period 1
        beta (float): discount factor
        r (float): return on savings
        Delta (float): income risk scale factor
        v2_interp (RegularGridInterpolator): interpolator for value function in period
 2

    Returns:

        (ndarray): value-of-choice

    """

    a = m1-d
    return w(a,d,r,Delta,v2_interp)
```

In [21]:

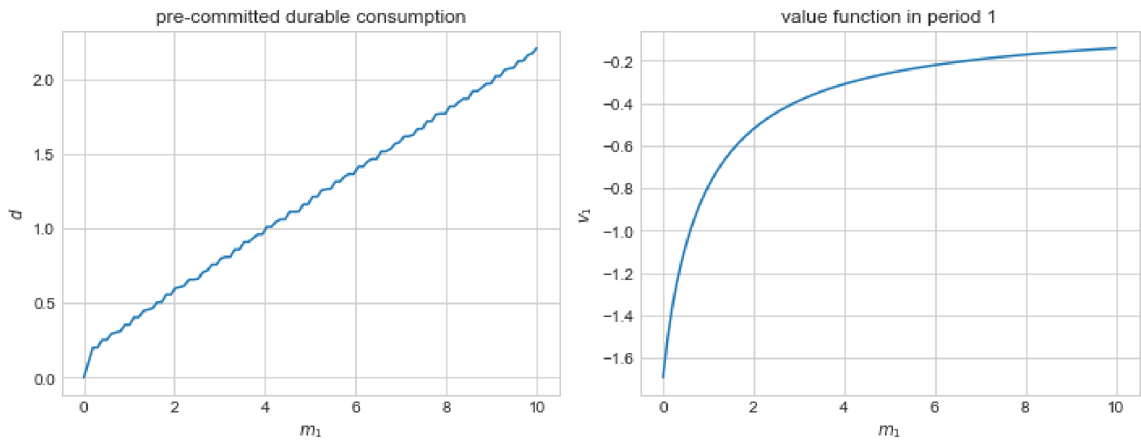
```
def solve_period_1(m1_vec,beta,r,Delta,v2_interp):
    """ post-decision value function in period 1

    Args:

        m1 (ndarray): vector cash-on-hand in the beginning of period 1
        beta (float): discount factor
        r (float): return on savings
        Delta (float): income risk scale factor
        v2_interp (RegularGridInterpolator): interpolator for value function in period
 2

    Returns:

        v1_vec (ndarray): value function in period 1
        d_ast_vec (ndarray): pre-commited durable consumption function

    """

    # a. grids
    v1_vec = np.empty(m1_vec.size)
    d_ast_vec = np.empty(m1_vec.size)

    # b. solve for each m1 in grid
    for i,m1 in enumerate(m1_vec):

        # i. objective
        obj = lambda x: -v1(x[0],m1,beta,r,Delta,v2_interp)

        # ii. initial guess (pre-commit half)
        x0 = m1*1/3

        # iii. optimize
        result = optimize.minimize(obj,[x0],method='L-BFGS-B',bounds=((1e-8,m1),))

        # iv. save
        v1_vec[i] = -result.fun
        d_ast_vec[i] = result.x

    return v1_vec,d_ast_vec
```

In [22]:

```
v1_vec,d_ast_vec = solve_period_1(m1_vec,beta,r,Delta,v2_interp)
```

In [23]:

```
fig = plt.figure(figsize=(10,4))
ax = fig.add_subplot(1,2,1)
ax.plot(m1_vec,d_ast_vec)
ax.set_xlabel('$m_1$')
ax.set_ylabel('$d$')
ax.set_title('pre-committed durable consumption')

ax = fig.add_subplot(1,2,2)
ax.plot(m1_vec,v1_vec)
ax.set_xlabel('$m_1$')
ax.set_ylabel('$v_1$')
ax.set_title('value function in period 1')
fig.tight_layout()
```



Question 3

In [24]:

```
Lambda = 0.2
m0_vec = np.linspace(1e-8,6,100)
d0_vec = np.linspace(1e-8,3,100)
```

In [25]:

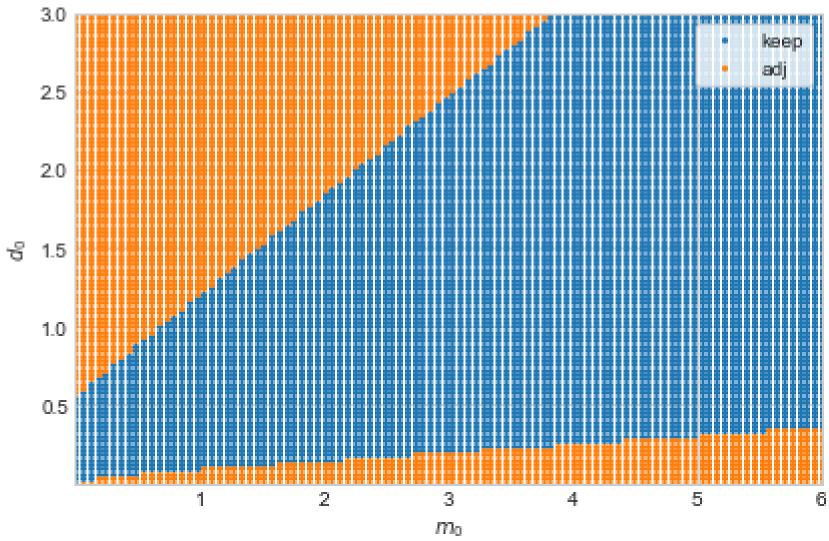
```
v1_interp = interpolate.RegularGridInterpolator([m1_vec], v1_vec,
                                                bounds_error=False,fill_value=None)
```

In [26]:

```
z = np.empty((m0_vec.size,d_vec.size))
for i,m0 in enumerate(m0_vec):
    for j,d0 in enumerate(d0_vec):

        # a. no adjustment
        v_keep = w(m0,d0,r,Delta,v2_interp)

        # b. adjustment
        v_adj = v1_interp([m0+(1-Lambda)*d0])[0]

        # c. best
        z[i,j] = 0 if v_keep > v_adj else 1
```

In [27]:

```
fig = plt.figure()
ax = fig.add_subplot(1,1,1)

m0g,d0g = np.meshgrid(m0_vec,d0_vec,indexing='ij')
ax.scatter(m0g[z == 0],d0g[z == 0],s=4,label='keep')
ax.scatter(m0g[z == 1],d0g[z == 1],s=4,label='adj')

ax.set_xlim([m0_vec[0],m0_vec[-1]])
ax.set_ylim([d0_vec[0],d0_vec[-1]])
ax.set_xlabel('$m_0$')
ax.set_ylabel('$d_0$')
ax.legend(frameon=True)
fig.tight_layout()
```



Gradient descent

Question 1

In [28]:

```python
def gradient_descent(f,x0,epsilon=1e-6,Theta=0.1,Delta=1e-8,max_iter=10_000):
    """ minimize function with gradient descent algorithm

    Args:

        f (callable): function
        x0 (np.ndarray): initial guess
        eps (float,optional): tolerance
        Theta (float,optional): initial step-size
        Delta (float,optional): step-size in numerical derivatives
        max_iter (int,optional): maximum number of iterations

    Returns:

        x (ndarray): minimum
        it (int): number of iterations used

    """

    # a. initialize
    x = x0
    fx = f(x0)

    # b. iterate
    it = 0
    while it < max_iter:

        # i. jacobian
        fp = np.empty(x0.size)
        for i in range(x0.size):
            x_ = x.copy()
            x_[i] = x[i] + Delta
            fp[i] = (f(x_)-fx)/Delta

        # ii. check convergence
        if np.max(np.abs(fp)) < epsilon: break

        # ii. line search
        theta = Theta
        while it < max_iter:

            # o. x value
            x_theta = x - theta*fp

            # oo. new function value
            fx_theta = f(x_theta)
            it += 1

            # ooo. break or continue line search
            if fx_theta < fx:
                fx = fx_theta
                x = x_theta
                break
            else:
                theta /= 2

    return x,it
```

In [29]:

```
def rosen(x):
    return (1.0-x[0])**2+2*(x[1]-x[0]**2)**2

x0 = np.array([1.1,1.1])
try:
    x,it = gradient_descent(rosen,x0)
    print(f'minimum found at ({x[0]:.4f},{x[1]:.4f}) after {it} iterations')
    assert np.allclose(x,[1,1])
except:
    print('not implemented yet')
```

```
minimum found at (1.0000,1.0000) after 331 iterations
```